Eigenstuff

Let A be an $n \times n$ matrix.

Suppose $Av = \lambda v$ for some nonzero $v \in \mathbb{R}^n$, $\lambda \in \mathbb{R}$.

Then v is called an eigenvector for A and λ is the eigenvalue corresponding to v.

Note: For now, eigenvalues are real. Later, we will consider complex eigenvalues.

Ex: $A = \begin{bmatrix} 16 \\ 52 \end{bmatrix}$ and $v = \begin{bmatrix} 6 \\ -5 \end{bmatrix}$

Then $Av = \begin{bmatrix} 16 \\ 52 \end{bmatrix}\begin{bmatrix} 6 \\ -5 \end{bmatrix} = \begin{bmatrix} -24 \\ 20 \end{bmatrix} = -4\begin{bmatrix} 6 \\ -5 \end{bmatrix} = -4v$

So $\begin{bmatrix} 6 \\ -5 \end{bmatrix}$ is an eigenvector with corresponding eigenvalue -4

Are there other eigenvectors corresponding to -4?

Yes! Any nonzero multiple of v is also an eigenvector of A:

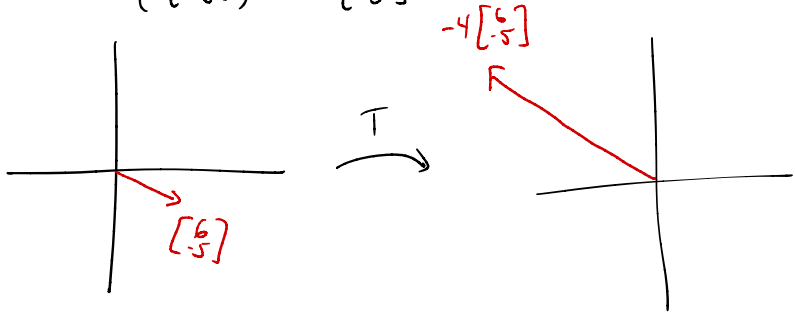
$$A(cv) = cAv = c(-4v) = -4(cv)$$

$\Rightarrow cv$ is also an eigenvector

Geometrically:

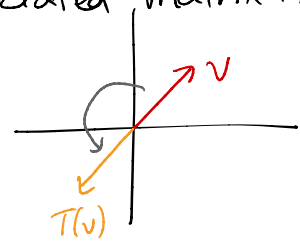
If $T: \mathbb{R}^n \to \mathbb{R}^n$ is the linear transformation given by $T(v) = Av$ and v is an eigenvector of A, then $T(v) = \lambda v \Rightarrow T$ scales v by λ

Ex: In the previous example where $A = \begin{bmatrix} 1 & 6 \\ 5 & 2 \end{bmatrix}$, we have $T\left(\begin{bmatrix} 6 \\ -5 \end{bmatrix}\right) = -4\begin{bmatrix} 6 \\ -5 \end{bmatrix}$



Ex: Let $T: \mathbb{R}^2 \to \mathbb{R}^2$ be the L.T. that rotates vectors 180°. What are the eigenvalues of the associated matrix A?

A 180° rotation negates every vector i.e. $Av = -v \quad \forall\, v \in \mathbb{R}^2$.



So -1 is an eigenvalue <u>and</u> every nonzero vector is an eigenvector corresponding to $\lambda = -1$.

So $\lambda = -1$ is the <u>only</u> eigenvalue.

Ex: Rotation by 90° has no (real) eigenvalues:
Av is not a multiple of v $\forall\, v \in \mathbb{R}^2$
$\Rightarrow$ v is not an eigenvector $\forall\, v \in \mathbb{R}^2$

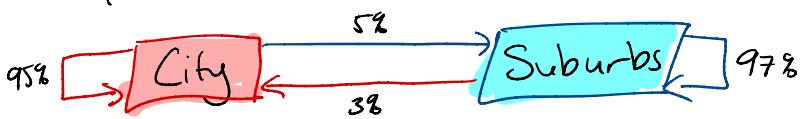
Important for Markov Chains:

If v is an eigenvector of A with eigenvalue λ, then v is an eigenvector of A^n with eigenvalue λ^n;

$$Av = \lambda v \implies A^n v = A^{n-1}(Av) = A^{n-1}(\lambda v) = \lambda A^{n-1} v = \cdots = \lambda^n v$$ (use induction)

Back to the Markov Chain example

Recall the population example described by



with Markov matrix $M = \begin{bmatrix} .95 & .03 \\ .05 & .97 \end{bmatrix}$ and initial vector $x_0 = \begin{bmatrix} .6 \\ .4 \end{bmatrix}$.

We'd like to determine $x_k = M^k x_0$ as $k \to \infty$.

Trick: express x_0 as a linear combination of eigenvectors.

First note: $M\begin{bmatrix} .6 \\ 1 \end{bmatrix} = \begin{bmatrix} .6 \\ 1 \end{bmatrix}$ and $M\begin{bmatrix} -1 \\ 1 \end{bmatrix} = .92\begin{bmatrix} -1 \\ 1 \end{bmatrix}$ (Check)

So $\begin{bmatrix} .6 \\ 1 \end{bmatrix}$ is an eigenvector corresp to eigenvalue 1

and $\begin{bmatrix} -1 \\ 1 \end{bmatrix}$ is an eigenvector corresp to eigenvalue .92

Next note: $x_0 = \begin{bmatrix} .6 \\ .4 \end{bmatrix} = .625\begin{bmatrix} .6 \\ 1 \end{bmatrix} - .225\begin{bmatrix} -1 \\ 1 \end{bmatrix}$ (To derive this, need to solve $\begin{bmatrix} .6 & -1 \\ 1 & 1 \end{bmatrix} x = \begin{bmatrix} .6 \\ .4 \end{bmatrix}$)

Hence $x_k = M^k x_0 = M^k\left(.625\begin{bmatrix} .6 \\ 1 \end{bmatrix}\right) + M^k\left(-.225\begin{bmatrix} -1 \\ 1 \end{bmatrix}\right)$

$= .625\, M^k \begin{bmatrix} .6 \\ 1 \end{bmatrix} - .225\, M^k \begin{bmatrix} -1 \\ 1 \end{bmatrix}$

$= .625\,(1)^k \begin{bmatrix} .6 \\ 1 \end{bmatrix} - .225\,(.92)^k \begin{bmatrix} -1 \\ 1 \end{bmatrix}$

$= \begin{bmatrix} .375 + (.225)(.92)^k \\ .625 - (.225)(.92)^k \end{bmatrix}$

Thus as $k \to \infty$, $x_k \longrightarrow \begin{bmatrix} .375 \\ .625 \end{bmatrix}$

So, in the long run, the populations will approach:

375,000 in city and 625,000 in suburbs.

More generally, since $\begin{bmatrix} .6 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \end{bmatrix}$ forms a basis for $\mathbb{R}^2$, we can do this for any initial state vector x_0

Let $x_0 = \begin{bmatrix} a \\ b \end{bmatrix}$ where $a+b=1$.

Then $x_0 = .625 \begin{bmatrix} .6 \\ 1 \end{bmatrix} - (.625a - .375b) \begin{bmatrix} -1 \\ 1 \end{bmatrix}$ (Solve $\begin{bmatrix} .6 & -1 \\ 1 & 1 \end{bmatrix} x = \begin{bmatrix} a \\ b \end{bmatrix}$ Keep in mind $a+b=1$)

Hence $x_k = M^k x_0 = .625 M^k \begin{bmatrix} .6 \\ 1 \end{bmatrix} - (.625a - .375b) M^k \begin{bmatrix} -1 \\ 1 \end{bmatrix}$

$= .625 (1)^k \begin{bmatrix} .6 \\ 1 \end{bmatrix} - (.625a - .375b)(.92)^k \begin{bmatrix} -1 \\ 1 \end{bmatrix}$

As $k \to \infty$, $x_k \longrightarrow \begin{bmatrix} .375 \\ .625 \end{bmatrix}$ again!

So the populations will approach 375,000 in city and 625,000 in suburbs regardless of initial populations

To understand why, we must:

- Discuss how to find eigenvalues and eigenvectors
- Determine when eigenvectors form a basis
- Develop the theory of Markov Chains